Fitting Function for Experimental Energy Ordered Spectra in Nuclear Continuum Studies

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Abstract
We review the main features of the Hk-EOS method for the experimental study of the nuclear continuum; the Hk technique, the Energy-ordering and the Energy Ordered Spectra EOS. The development of a method for computing the fitting function is presented an tested for both cases, spin-independent and spin-dependent. The simulated spectra are fitted well by the developed function. Applications of the fitting function for some experimental situations are discussed, which includes the fit to spectra considering a non-ideal response function of the detector array.

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The study of the nuclear continuum consists in the research of the nucleus at states with high energy and high spin, where the high level density enables us to talk about a continuum of states. The Hk-EOS method [1] has been developed, this new experimental method studies the nuclear continuum by analyzing the gamma decay of excited states. Hk-EOS mixes the Hk technique [2] and the Energy-ordering [3] for producing the Energy Ordered Spectra EOS, from which we can extract physical parameters describing the nucleus [4]. In order to achieve that, a fitting function is necessary. Previous developments [5, 6] have used Order Statistics [7] and Extreme Value Theory [8] to compute the fitting function for the EOS, but the results have not been satisfactory.

The entire Hk-EOS method has been simulated obtaining good results [1], however we need a fitting function for a complete analysis of the obtained spectra.

## 2 Theoretical background

The nuclear states at high energy and spin define the nuclear continuum. At this energy the spacing between adjacent states is very small and due to the line width, they overlap, therefore we do not talk about discrete states, instead we talk about the number of states per energy unit which gives us a continuous level density. In order to develop an experimental method to study such states we have to use certain physical concepts.

### 2.1 Yrast states and collective rotational bands

In physical systems (for instance the rigid body in classical mechanics) there is a minimum energy for a given angular momentum. In the nuclear case, for each nuclear spin $I$ a physical state with minimum energy $E_{\text{yrast}}(I)$ exists. Those states $(I, E_{\text{yrast}}(I))$ are yrast states.

The set of yrast states is called yrast line and defines the intrinsic excitation energy $U$, which is the energy relative to the yrast state at the same $I$;

$$U = E - E_{\text{yrast}}(I).$$
2.2 Gamma strength

There are excited states above the yrast line, they can be due to single particle excitations or due to the collective rotation of nucleons around a common axis. When different excited states are due to a rotation about the same axis those sates belong to the same collective rotational band. As we will show later it is a very important feature of the nuclear continuum structure because there are transitions between states of these bands.

The arrangement of rotational bands is similar to the yrast line. As we can see in the phase space plot Fig. 1, the collective rotational bands are approximately parallel to the yrast line.

2.2 Gamma strength

In quantum mechanics the transition probability between two states of a system is given by the Fermi’s Gold Rule, where we have to compute the matrix element of the quantum operator $\hat{H}$ of the interaction that produces the transition.

$$|\langle f | \hat{H} | i \rangle|^2.$$ 

For gamma transitions that matrix element is known as the gamma strength. It gives the probability to go from the initial state $|i\rangle$ to a final state $|f\rangle$ with the emission of a $\gamma$-ray. The gamma strength will depend on the energy of the transition $E_\gamma$ ($\gamma$-ray’s energy) and the character and multipolarity of the radiation.

The Weisskopft’s approach \[9\] uses the single-particle approximation, obtaining a gamma strength given by:

$$\Gamma(O\lambda : E_\gamma) = C_{O\lambda}E_\gamma^{2\lambda+1}$$

where $O$ is the character of the radiation and $\lambda$ its multipolarity, $C$ is a constant that depends on $O$ and $\lambda$. For instance for electric dipolar radiation we obtain:

$$\Gamma(E1 : E_\gamma) = C_{E1}E_\gamma^3.$$ (1)
A more accurate gamma strength functional relation is the Giant Dipole Resonance (GDR) which takes into account collective effects of the nucleons. For electric dipolar radiation its mathematical form is given by \[10\]:

\[
\Gamma_{\text{GDR}}(E_1: E_\gamma) = \frac{\sigma}{3(\pi\hbar c)^2} E^4 \sum_{k=1}^{k_{\text{def}}} \frac{w_k \Gamma_{Gk}}{(E_{Gk}^2 - E_\gamma^2)^2 + \Gamma_{Gk}^2 E_{Gk}^2} \]

where the parameters \(w_k, E_{Gk}, \Gamma_{Gk}\) are related to peaks in the gamma strength due to \(k_{\text{def}}\) nuclear deformation axis, and \(\sigma\) is the GDR peak cross section.

### 2.3 Level density

The nuclear level density is a very important quantity, besides it is one of the main targets of the Hk-EOS method \([1]\) for experimental study. The nucleus can be modeled as a Fermi Gas made up by protons and neutrons, in order to compute the level density under this assumption, there are different mathematical treatments.

The simplest one is the Constant Temperature (CT) level density \([11]\), that using thermodynamical arguments sets:

\[
\rho(U) = \rho(0)e^{U/T},
\]

where \(T\) is the nuclear temperature and \(\rho(0)\) is a constant. Notice that the level density depends explicitly and only on the intrinsic excitation energy \(U\).

An important result from the statistical mechanics is the Lang’s level density \([12]\) for the Fermi Gas (we will refer as FG). It sets the level density as:

\[
\rho(U, I) = \sqrt{a} \left( \frac{\hbar^2}{2J} \right)^{3/2} (2I + 1) \frac{e^{2\sqrt{aU}}}{(U + \frac{3}{2}T)^2} \]

where \(a\) is the level density parameter and the nuclear temperature is given by

\[
T = \sqrt{\frac{U}{a} + \left( \frac{3}{4a} \right)^2 + \frac{3}{4a}},
\]

notice that the Lang’s level density depends on \(U\) and the spin \(I\).

### 2.4 Probability of gamma emission

As we stated before the transition probability is given by Fermi’s Gold Rule, where in addition to the gamma strength we have to consider the level density at the initial and final state. Then the probability \(p_{i\rightarrow f}\) for an E1 transition to take place from an initial phase space differential region with intrinsic excitation energy \(U_i\) and spin \(I_i\) to a final one with \(U_f\) and \(I_f\) is proportional to

\[
p_{i\rightarrow f} \propto \Gamma(E_1; E_\gamma) \frac{\rho(U_f, I_f)}{\rho(U_i, I_i)}.
\]
2.5 GAMBLE

GAMBLE [13] is a powerful and recognized simulation code for the decay of states in the nuclear continuum; it considers all the known physical processes and variables present in the continuum gamma decay. GAMBLE provides a lot of options for modeling the nuclear structure, as the yrast states and the level density. Furthermore, it allows to set the gamma strength for the transitions and to fix the entry populations for the decay process.

3 Hk-EOS method

In [1, 4] a new experimental method in the framework of the so-called nuclear continuum spectroscopy has been developed. In particular, the Hk-EOS method achieves this, studying the gamma decay of states in the continuum. Next, we will expose the main features of the method and the role of the fitting function, which is the main purpose of the present work.

3.1 The experiment

Experimental setup as the detector arrays AGATA, GAMASPHHERE and GASP is characterized by its approximately $4\pi$ coverage and its high segmentation (Fig. 2); those detector arrays can be used for the Hk-EOS method application.

The experiment consists in producing states at high energy and spin (entry states), which can be done by fusion-evaporation reactions. Next, the excited nucleus will decay to reach the ground state via gamma transitions (Fig. 2). Each nucleus will make a set of transitions and therefore it will emit a series of $\gamma$-rays, this is a cascade. The multiplicity $M$ is the number of $\gamma$-rays in a cascade, it is a random quantity because of the probabilistic nature of the decay process.

The above experimental setup allows to detect each photon of a cascade. Because of the short lifetime of the states occupied during the decay process, the electronics available for the signal processing of the detectors is unable to distinguish the temporal ordering of the emission of the $\gamma$-rays. Then in the experiment, we can know the energy of all the $\gamma$-rays of each cascade, but
3.2 Energy-ordering. TOS and EOS

Figure 3: Simulated TOS and EOS spectra. The two spectra are different in the entire energy region, however at high energy they are similar [1].

we cannot know which of them was emitted first (primary radiation).

3.2 Energy-ordering. TOS and EOS

Each cascade is made up by a series of $\gamma$-rays whose energies can be labeled as:

$$E_1 \quad E_2 \quad E_3 \quad \ldots \quad E_M,$$

the $\gamma$-energies are ordered as they were emitted, in this way $E_1$ is the energy of the first $\gamma$-ray, $E_2$ is the energy of the second $\gamma$-ray and so on (Fig. 2). This defines a temporal sequence and therefore a Time-ordering.

Using the Time-ordering we can define the Temporal Ordered Spectra TOS. The TOS $N$ spectrum is constructed with the $E_N$ $\gamma$-energy of each cascade. Hence TOS $N = 1$ is the spectrum made up by the first $\gamma$-ray of each cascade (primary radiation spectrum), TOS $N = 2$ is the spectrum made up by the second $\gamma$-ray of each cascade and so on.

Notice that assuming known the entry states distribution, the TOS $N = 1$ spectral shape is given by eq. (5); because we know the entry states, we can calculate the probability for a transition with energy $E_1$ integrating eq. (5) over the entry states [4]. So the TOS $N = 1$ fitting function is already known.

As we stated before the Time-ordering is not accessible experimentally therefore the TOS cannot be constructed for a real experiment. The Energy-ordering [2] has been proposed as an approximation to the Time-ordering, it orders the $\gamma$-energies by their value, taking first the most energetic $\gamma$-ray, second the second most energetic and so on. As in Time-ordering, we can define the Energy Ordered Spectra EOS $N$; the EOS $N = 1$ is the spectrum constructed with the most energetic $\gamma$-ray of each cascade, the EOS $N = 2$ is constructed with the second most energetic $\gamma$-ray of each cascade and so on.
In a previous work \cite{14} the notations TOS \( N \) and EOS \( N \) have been used since an analysis of the spectra for different ordinal \( N \) has been presented. But here, because the TOS \( N = 1 \) and EOS \( N = 1 \) are the only TOS and EOS that we use, we refer the TOS \( N = 1 \) and EOS \( N = 1 \) only as TOS and EOS respectively.

The Energy-ordering has been proposed as an approximation to the Time-ordering under the assumption that the most energetic transition may take place first. Now the question is how well statistically speaking the highest energetic transition represents the primary radiation, that means if the TOS and EOS are similar enough; the answer to this question is fundamental to establish if we can fit the EOS (experimentally accessible) with eq. (5).

The TOS and EOS were constructed from a GAMBLE simulation. In Fig. 3 we can see that the two spectra are not equal, that means that we cannot use the eq. (5) to fit the EOS in the entire energy range. We need to develop a fitting function, a mathematical expression to fit the EOS in the entire energy range, using the physical models condensed in eq. (5). However the TOS and EOS are very similar at high energies (Fig. 3), this fact has been used in previous work \cite{4} to extract physical parameters from the EOS.

### 3.3 Hk technique

The Hk technique \cite{3} allows to isolate cascades coming from a small region of the phase-space. It is necessary for a differential study of the nuclear structure since the entry states produced after fusion-evaporation reactions have a wide distribution over energy and spin.

Hk works as follows: Suppose that in the real experiment a cascade fires \( k \) detectors producing an energy pulse \( h_i \) in each one. Then the total energy \( H \) deposited in the entire detection system is

\[
H = \sum_{i=1}^{k} h_i.
\]

There is a relation between the pair \((k, H)\) and \( M \) of the cascade and the initial excitation energy \( E \) of the nucleus that emitted it. However due to the detection process (Compton scattering), the non-ideal response function and the finite segmentation of the detector array, the relation \( (M, E) \rightarrow (k, H) \) is not a deterministic one, it is a probabilistic one. To a pair \((M, E)\) corresponds a probability cloud on the \((k, H)\) space, because there is a response function of the entire detector array:

\[
(M, E) \rightarrow (k, H)
\]

response function.

If the response function is known the deconvolution process can be performed; a pair \((H, k)\) produces a probability cloud on \((E, M)\) space:

\[
(k, H) \rightarrow (M, E)
\]

deconvolution.

Furthermore a relation between the multiplicity \( M \) and the spin \( I \) where the cascade was originated has been observed \cite{1}, therefore we can set a probabilistic relation

\[
(k, H) \rightarrow (I, E),
\]

(6)
that means that given a pair \((k, H)\), there is a probability distribution on the \((I, E)\) space for the initial states which could produce a cascade recorded with \(H\) and \(k\) by the detection system.

Eq. (6) has been studied for different detector arrays [1] using simulations of the entire detection system. It was shown that eq. (6) can be synthesized for a real experiment, and that the contour curves of the distribution correspond to ellipses in the \((I, E)\) space (Fig. 4). Thus setting a gate on \(H\) and \(k\) in the experiment, that is, isolating the cascades detected with \(k\) and with sum of energies between \(H\) and \(H + \Delta H\), we are able to construct the EOS from a known small region of the phase space.

4 Fitting function

4.1 Continuum Gamma decay

Once we have populated states in the continuum, those states will decay by gamma transitions. There are two kinds of gamma transitions [13]:

- Collective transitions: In the case of deformed nuclei, these are transitions inside the same collective rotational band, therefore they have low fixed energy, \(E_\gamma < 2\) MeV. Because the bands cross approximately parallel to the yrast line those transitions do not change the intrinsic excitation energy (do not depend on \(\rho(U, I)\)), but reduce the spin \(I\).

- Statistical transitions: Transitions between single particle excitation or collective (different bands) states whose energies are not fixed, they have a statistical distribution that depends on the level density \(\rho(U, I)\) of the initial and final states. These transitions reduce the intrinsic excitation energy \(U\) and are very energetic, \(E_\gamma > 2\) MeV.

Each one of those transitions may occur by different electromagnetic modes, the statistical transitions can be E1, E2 or M1 (E1, E2, and M1 in Fig. 5); the collective transitions may be E2 and M1 + E2 (E2c and E2/M1 in Fig. 5).

The decay process is a statistical one where the kind of transition and its features (energy and electromagnetic mode) are random variables, therefore to model the EOS is a complex task. Because we are interested in the study of nuclear properties like the level density we need to look for the statistical transitions, but in the real experiment we will obtain the entire spectrum (all kind of transitions). Hence the objectives are to find a method to isolate statistical transitions and to model the EOS in order to develop a fitting function.
4.2 Spin-independent case

In the spin-independent case [4] the nuclear state is characterized only by its intrinsic excitation energy $U$ (no spin), hence the level density and the transition probability depends only on $U$ and $E_\gamma$. Eq. (5) turns into

$$p_{f\rightarrow i} \propto \Gamma(E1; E_\gamma) \frac{\rho(U_f)}{\rho(U_i)}.$$
Assuming an initial excitation energy $U_i = U$ the final excitation energy is $U_f = U - E_\gamma$. Therefore the transition probability is proportional to

$$\Gamma(E_1; E_\gamma) \frac{\rho(U_f)}{\rho(U)} = \Gamma(E_1; E_\gamma) \frac{\rho(U - E_\gamma)}{\rho(U)},$$

where

$$\rho(U) \propto e^{\frac{U}{T}},$$

$$\rho(U) \propto \frac{e^{\frac{2\sqrt{aU}}{U + \frac{3}{2}T}}}{(U + \frac{3}{2}T)^2}$$

for the Constant Temperature and the Fermi Gas level density respectively.

Therefore given an initial excitation energy $U$ the probability $p(E_\gamma, U)$ for a transition to take place with energy $E_\gamma$ can be written in the form

$$p(E_\gamma, U) = \frac{1}{F(U)} \Gamma(E_1; E_\gamma) \rho(U - E_\gamma) = \frac{f(E_\gamma, U)}{F(U)}.$$

For convenience we have defined

$$f(E_\gamma, U) = \Gamma(E_1; E_\gamma) \rho(U - E_\gamma)$$

and $F(U)$ as a normalization factor given by

$$F(U) = \int_0^U f(E_\gamma, U) dE_\gamma.$$

In the spin-independent case we only consider the E1 statistical transitions, the collective transitions are not considered because they do not contribute to the EOS because they do not change $U$. Therefore a nucleus with initial $U$ decays via gamma transitions whose energies are

Figure 6: GAMBLE simulation of the EOS for statistical E1 transitions; total spectrum and its decomposition in the different $\Delta I$ [14].
random variables with a probability distribution given by eq. (9). The process ends when a zero intrinsic excitation energy is reached.

To calculate the EOS fitting function we take a nucleus with a given initial excitation energy $U$. It will decay, emitting a temporal sequence of $\gamma$-rays with energies $E_1, E_2, E_3, \ldots$, each of them taken from a probability density function as eq. (9). But now the intrinsic excitation energy $U_i$ for the $i$-transition reflects the decay process up to that point, because it depends on the energies of the previous $\gamma$-rays. Then the probability density function is

$$p(E_i, U_i) = \frac{f(E_i, U_i)}{F(U_i)},$$

(12)

where

$$U_i = U - \sum_{j=1}^{i-1} E_j.$$  

(13)

The nucleus cannot decay to intrinsic excitation energies below $U = 0$ (ground state), hence the following condition must be fulfilled:

$$E_i < U_i.$$

Taking into account that the first $\gamma$-ray can have an energy between 0 and $U$, the second one between 0 and $U - E_1$, the third one between 0 and $U - E_1 - E_2$, and so on; we can see that the energy range for a $\gamma$-ray is larger (to high energies) for those emitted first. Because of this reason the probability to find the $i$th $\gamma$-ray as the most energetic is higher for the first ones. Therefore we will assume that the most energetic $\gamma$-ray must be one of the first $n$ $\gamma$-rays, and in order to calculate the EOS fitting function $p_{N_\gamma=1}(E_\gamma, U)$ only they will be considered. This is not an exact calculation, but as we will show next, $n$ allows us to fix well the fitting function in a desired energy range.

### 4.2.1 $n = 1$

Taking $n = 1$ we consider only the first $\gamma$-ray, therefore it has to be the most energetic. Thereby $p_{N_\gamma=1}(E_\gamma, U)$ is the same probability density function that $p(E_1, U_1)$, hence

$$p_{N_\gamma=1}(E_\gamma, U) = \frac{f(E_\gamma, U)}{F(U)}.$$

### 4.2.2 $n = 2$

With $n = 2$ we have to consider only the first two $\gamma$-rays, then there are two possibilities, either the first emitted or the second emitted $\gamma$-ray being the most energetic. We call $P_1(E_\gamma, U)$ the probability of the first $\gamma$-ray being the most energetic with an energy $E_\gamma (= E_1 > E_2)$. Correspondingly the other case gives $P_2(E_\gamma, U)$ as the probability of the second $\gamma$-ray being the most energetic with an energy $E_\gamma (= E_2 > E_1)$.

In order to obtain the first $\gamma$-ray with energy $E_\gamma$ we need two things to happen; to obtain $E_\gamma$ for the first $\gamma$-ray, and to obtain an $E_2$ less than $E_\gamma$ for the second $\gamma$-ray. $P_1(E_\gamma, U)$ can
Figure 7: Scheme of energy levels and transition energies for the first two $\gamma$-rays when a) the first one or b) the second one is the most energetic.

Be calculated as the multiplication of the probability of the two events, $P_{E_1=E_\gamma}$ and $P_{E_2<E_\gamma}$ respectively:

$$P_1(E_\gamma, U) = P_{E_1=E_\gamma} \times P_{E_2<E_\gamma}.\tag{12}$$

Using eq. (12) and the fact that the maximum energy for the second $\gamma$-ray is $U_2 = U - E_1$ those probabilities can be calculated as

$$P_1(E_\gamma, U) = \frac{f(E_\gamma, U)}{F(U)} \times \int_0^{\min\{E_\gamma, U-E_\gamma\}} dE_2 \frac{f(E_2, U-E_\gamma)}{F(U-E_\gamma)},\tag{14}$$

We have to integrate over $E_2$, the upper limit in the integral appears because $E_2$ has two possibles uppers limits, $E_1 = E_\gamma$ because we have imposed that $E_1 > E_2$; or $U_1 = U - E_1$ because the maximum energy for a transition is the intrinsic excitation energy for that transition (Fig. 7a). We have to choose the minimum of the two values depending on $E_\gamma$.

For $P_2(E_\gamma)$ we need the probability of the first $\gamma$-ray with an energy lower than $E_\gamma$ and the probability for the second $\gamma$-ray with energy $E_\gamma$

$$P_2(E_\gamma, U) = P_{E_1<E_\gamma} \times P_{E_2=E_\gamma}.\tag{15}$$

Using eq. (12) and taking into account that because we have fixed the energy of the second $\gamma$-ray $E_2$, $E_1$ has a new upper limit. We have the condition (Fig. 7b)

$$E_2 < U_2 = U - E_1,$$

however we have fixed the energy of the second $\gamma$-ray as $E_\gamma$, hence the previous condition turns to a condition for $E_1$:

$$E_1 < U - E_\gamma,$$

which sets an upper limit for $E_1$.

Using eq. (12) and the previous condition, $P_2(E_\gamma, U)$ can be calculated as

$$P_2(E_\gamma, U) = \int_0^{\min\{E_\gamma, U-E_\gamma\}} dE_1 \frac{f(E_1, U)}{F(U)} \times \frac{f(E_\gamma, U-E_1)}{F(U-E_1)}.$$

(15)
Finally, to calculate the fitting function we have to add the two contributions since both cases produce $E_\gamma$ as the energy of the most energetic $\gamma$-ray,

$$p_{N_E=1}(E_\gamma, U) = P_1(E_\gamma, U) + P_2(E_\gamma, U). \quad (16)$$

The upper limit of the integral in eq. (14) and eq. (15) produces different behaviors on different energy ranges. If $\min(E_\gamma, U - E_\gamma) = E_\gamma$, we have then

$$E_\gamma < U - E_\gamma,$$

$$2E_\gamma < U,$$

$$E_\gamma < U/2,$$

and if $\min(E_\gamma, U - E_\gamma) = U - E_\gamma$ we get $E_\gamma > U/2$. These two inequalities define two different regimes.

Further, an important result is that for $E_\gamma > U/2$ it is possible to obtain a closed expression for $P_1(E_\gamma, U)$:

$$P_1(E_\gamma, U) = \frac{f(E_\gamma, U)}{F(U)} \int_0^{U-E_\gamma} dE_2 \frac{f(E_2, U - E_\gamma)}{F(U - E_\gamma)} = \frac{f(E_\gamma, U)}{F(U)} \frac{f(U - E_\gamma)}{F(U - E_\gamma)} = \frac{f(E_\gamma, U)}{F(U)},$$

which is the probability density function for the first $\gamma$-ray, this is the reason why EOS and TOS are similar at high energies, as we stated before.

### 4.2.3 $n = 3$

Considering only the first three $\gamma$-rays, the probability to obtain the most energetic with energy $E_\gamma$ is:

$$p_{N_E=1}(E_\gamma, U) = P_1(E_\gamma, U) + P_2(E_\gamma, U) + P_3(E_\gamma, U). \quad (17)$$

where $P_i(E_\gamma, U)$ is the probability of the $i$th $\gamma$-ray being the most energetic with a energy $E_\gamma$.

The probability $P_1(E_\gamma, U)$ is the multiplication of the probability to obtain $E_1 = E_\gamma$, and the probability to obtain an energy for the second and third $\gamma$-ray less than $E_\gamma$,

$$P_1(E_\gamma, U) = P_{E_1=E_\gamma} \times P_{E_2 < E_\gamma} \times P_{E_3 < E_\gamma}.$$

Using eq. (12) and taking into account that $E_2$ and $E_3$ are limited by $U - E_1$ and $U - E_1 - E_2$ respectively $P_1(E_\gamma, U)$ can be computed as:

$$P_1(E_\gamma, U) = \frac{f(E_\gamma, U)}{F(U)} \times \int_0^{\min(E_\gamma, U-E_\gamma)} dE_2 \frac{f(E_2, U - E_\gamma)}{F(U - E_\gamma)} \times \int_0^{\min(E_\gamma, U-E_\gamma-E_2)} dE_3 \frac{f(E_3, U - E_\gamma - E_2)}{F(U - E_\gamma - E_2)}. \quad (18)$$

In order to calculate $P_2(E_\gamma, U)$ we have to consider that $E_2 = E_\gamma$, $E_3$ is bounded by $E_\gamma$ and $U - E_1 - E_2$, because we have fixed $E_2$ the condition $E_2 < U - E_1$ turns to $E_1 < U - E_2$ hence
$E_1$ is limited by $E_\gamma$ and $U - E_\gamma$. Then $P_2(E_\gamma, U)$ is

$$P_2(E_\gamma, U) = \int_0^{\min\{E_\gamma, U - E_\gamma\}} \frac{f(E_1, U)}{F(U)} \times \frac{f(E_\gamma, U - E_1)}{F(U - E_1)} \times \int_0^{\min\{E_\gamma, U - E_\gamma - E_1\}} \frac{f(E_3, U - E_\gamma - E_1)}{F(U - E_\gamma - E_1)} \, dE_3. \tag{19}$$

For $P_3(E_\gamma, U)$ we have to consider again the condition $E_2 < U - E_1$ which means $E_1 < U - E_2$; we also need $E_3 < U - E_1 - E_2$, hence $E_2 < U - E_3 - E_1$; and because the third $\gamma$-ray is the most energetic now, we have to impose $E_3 = E_\gamma$. Thus we obtain

$$P_3(E_\gamma, U) = \int_0^{\min\{E_\gamma, U - E_\gamma\}} \frac{f(E_1, U)}{F(U)} \times \int_0^{\min\{E_\gamma, U - E_\gamma - E_1\}} \frac{f(E_2, U - E_1)}{F(U - E_1)} \times \frac{f(E_\gamma, U - E_1 - E_2)}{F(U - E_1 - E_2)} \, dE_2. \tag{20}$$

Studying the integral's upper limits of eq. (18, 19, 20) it can be shown that there are three regimes:

$$0 < E_\gamma < U/3$$

$$U/3 < E_\gamma < U/2$$

$$U/2 < E_\gamma < U.$$

4.2.4 $n$

The expressions for the $n = 1, 2, 3$ cases can be generalized to any number $n$. In this case the fitting function is the sum of $n$ contributions $P_i(E_\gamma, U)$ which are the probability to obtain the $i$th $\gamma$-ray as the most energetic with energy $E_\gamma$.

$$P_i(E_\gamma, U) = \prod_{j=1; j \neq i}^n \int_0^{\min\{E, U - E - \sum_{k=1; k \neq i}^{j-1} E_k\}} \frac{f(E_j, U - \sum_{k=1}^{j-1} E_k)}{F(U - \sum_{k=1}^{j-1} E_k)} \times \frac{f(E, U - \sum_{k=1}^{j-1} E_k)}{F(U - \sum_{k=1}^{j-1} E_k)} \, dE_j, \tag{21}$$

$$p_{NE=1}(E) = \sum_{i=1}^n P_i(E), \tag{22}$$

once the eq. (21) has been expanded, $E_i$ must be replaced by $E_\gamma$.

From the upper limits on the integrals in eq. (21), we can show that there are different regimes for energy intervals of the form:

$$\frac{U}{m + 1} < E_\gamma < \frac{U}{m},$$

with $m = 1, 2, 3, \ldots, n$. This feature of the Order Statistics of a bound variable has been used before in previous attempts [5] to calculate the fitting function.
4.2 Spin-independent case

Figure 8: Monte Carlo simulation for the spin-independent case of the $^{170}$Hf decay, using the Constant Temperature ($T = 0.74$ MeV) level density, the Weisskopf gamma strength and an initial excitation energy $U = 11$ MeV. The fitting function $p_{N\gamma=1}$ was calculated for $n = 3, 4, 5, 6$. The fit is good for high energy and improves at low energy with higher $n$.

4.2.5 Testing the fitting function

In order to check the developed fitting function eq. (21) and eq. (22) we performed a Monte Carlo simulation of the gamma continuum decay for the spin-independent case. The simulation works as follows:

- The intrinsic excitation energy is set as $U$.
- The energy of the first $\gamma$-ray $E_1$ is taken from the probability density function eq. (12) with an initial excitation energy $U$.
- The new intrinsic excitation energy is calculated as $U - E_1$. 
4.2 Spin-independent case

- The energy of the second $\gamma$-ray $E_2$ is taken from the probability density function eq. (12) with an initial excitation energy $U - E_1$.

- The process continues in this way until a low ($\approx 1$MeV) intrinsic excitation energy is reached.

- The higher $\gamma$-ray energy is selected and stored to construct the histogram.

- The process is repeated for each cascade.

All the simulated spectra were normalized with its integral equal to one, because the fitting function was calculated as a probability density function normalized in that way.

We simulated the $^{170}$Hf decay using two different models for the level density and the gamma strength. In the first case (CT-W) we use the Constant Temperature level density eq. (3) and the Weisskopf’s gamma strength for dipolar radiation eq. (1), hence the function $f(E_i, U_i)$ in eq. (12) is:

$$ f(E_i, U_i) = E_i^3 e^{-E_i/T}. $$

Notice that it is a simple relation, it does not depend on $U_i$ and it can be integrated analytically to obtain $F(U_i)$. We set the initial excitation energy at $U = 11$ MeV, with this value the nucleus under study is described by a nuclear temperature $T = 0.74$ MeV [5].

The fitting function eq. (21) had to be numerically computed, due to the large number of integrals that cannot be calculated analytically. We performed the simulation and calculated the fitting function for many $n$. In Fig. 8 we can observe that the fit is good at high energy for all the cases, and that taking a higher $n$ the fit improves at lower energies. In this case we obtain a visually perfect fit in the entire energy range taking $n = 6$.

We performed a Monte Carlo simulation (FG-GDR) using the Fermi Gas level density eq. (4) and the Giant Dipole Resonance eq. (2). The parameters describing the $^{170}$Hf decay are listed in Table 1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Level density</strong></td>
<td></td>
</tr>
<tr>
<td>$a$ (MeV$^{-1}$)</td>
<td>20.</td>
</tr>
<tr>
<td>$GDR$</td>
<td>$k = 1$ $k = 2$</td>
</tr>
<tr>
<td>$w_k$</td>
<td>0.4 0.6</td>
</tr>
<tr>
<td>$E_{Gk}$ (MeV)</td>
<td>11.9 15.2</td>
</tr>
<tr>
<td>$\Gamma_{Gk}$ (MeV)</td>
<td>2.8 4.7</td>
</tr>
</tbody>
</table>

Table 1: Parameters describing the Hf$^{170}$ continuum gamma decay [4]. They were used in the spin-independent simulation.

In this case the function to be used in eq. (12) is:

$$ f(E_i, U_i) = \frac{e^{2\sqrt{a(U_i - E_i)}}}{(U_i - E_i + \frac{3}{2}T(U_i - E_i))^2} E_i^4 \sum_{k=1}^{2} \frac{w_k \Gamma_{Gk}}{E_i^2 - E_{Gk}^2 + \Gamma_{Gk}^2 E_i^2}. $$
4.2 Spin-independent case

Figure 9: Monte Carlo simulation for the spin-independent case of the $^{170}$Hf decay, using the Fermi Gas level density, the Giant Dipole Resonance gamma strength with the parameters in Table 1 and an initial excitation energy $U = 11$ MeV. The fitting function $p_{N_E=1}$ was calculated for $n = 2, 3, 4, 5$. The fit is good for high energy and improves at low energy with higher $n$.

notice that it is a complex relation that depends on $U_i$, this function cannot be integrated analytically to obtain $F(U_i)$.

As in the previous case we performed the simulation and calculated the fitting function for different $n$ values. In Fig. 9 we can see again a good fit at high energy, it improves at low energies when $n$ is increased. In this case we obtain a perfect fit in the entire energy range with $n = 5$.

In order to understand why the fitting function works well, we calculated the average energies for the first six $\gamma$-rays. First we computed the average energy for the first gamma ray $\langle E_1 \rangle$ using eq. (12), with this value we set the intrinsic excitation energy for the second transition $U_2 = U - \langle E_1 \rangle$ and calculated the second $\gamma$-energy, with this value we set the third transition’s excitation energy and so on.
Figure 10: Average energies $\langle E_i \rangle$ for the first six $\gamma$-rays, for the two studied cases; CT-W and FG-GDR.

The average energies for the two studied cases are shown in Fig. [10]. As expected, the average energy is higher for the first $\gamma$-ray and decreases for the following, likewise the probability of being the most energetic. For FG-GDR the average energy decreases quickly, almost linearly; while for CT-W the decrease is slow, for the first three $\gamma$-rays the average energies are nearly equal. This explain why for FG-GDR the fitting function converges faster than for CT-W.

The developed fitting function works well for the spin independent case, choosing an appropriated $n$ the EOS can be fitted in the entire energy range. This means that the fitting function allows us to distinguish between different nuclear models (level density and gamma strength) since spectral shape depends on the models [14].

4.3 Spin-dependent case (Real case)

By spin-dependent we understand the real case where all kinds of transitions can take place and the spin plays an important role to determine the probability to obtain a given transition. Now the probability for an E1 transition will depend on the spin of the initial and final states,

$$ p(E_{\gamma}) \propto \Gamma(E_{1}; E_{\gamma}) \frac{\rho(U_f, I_f)}{\rho(U_i, I_i)}. \quad (23) $$

We have to define the effective initial excitation energy $U^{\text{eff}}$ for a transition. Suppose an initial nuclear state with energy $E$ and spin $I$, such state has an intrinsic excitation energy

$$ U = E - E_{\text{yrast}}(I). $$

Depending on the change in the spin $\Delta I$ the $\gamma$-energy could vary in certain energy range, not necessarily between 0 and $U$ as in the spin-independent case. The maximum possible energy for the transition is $U^{\text{eff}}$, which is:

$$ U^{\text{eff}} = E - E_{\text{yrast}}(I + \Delta I) = U + E_{\text{yrast}}(I) - E_{\text{yrast}}(I + \Delta I). \quad (24) $$
It is the initial intrinsic excitation energy that must be used in eq. (12) to calculate the energy of the $\gamma$-ray of the transition.

To calculate the EOS fitting function for a well defined initial state $(I, E)$, fixed energy and spin, important assumptions and approximations have to be done:

1. The first transition is statistical $E1$ $\Delta I = -1$, for this reason we will have an effective initial excitation energy $U_{eff} = U + E_{yrast}(I) - E_{yrast}(I - 1)$ for this transition.

2. It is followed by:
   - A set of collective transitions that carry the nucleus to low spin states where $U_{eff} \approx U$ because $E_{yrast}(I) \approx E_{yrast}(I + \Delta I)$, or
   - it is followed by statistical transitions with $\Delta I = -1, 0, 1$, which means that on average we have $U_{eff} \approx U$ for those transitions.

Taking into account the previous assumptions we will postulate that the spin-dependent fitting function $p_{SE}^S$ is the spin-independent function $p_{NE}^S$, with an initial intrinsic excitation energy given by the $U_{eff}$ for the first $\Delta I = -1$ transition, eq. (24). The fitting function for a well defined state $(I, E)$ is:

$$p_{SE=1}^S(E_\gamma; I, E) = p_{NE=1}(E_\gamma, E - E_{yrast}(I - 1)),$$

or in terms of the intrinsic excitation energy, for an initial state $(I, U)$:

$$p_{NE=1}^S(E_\gamma; U, I) = p_{NE=1}(E_\gamma, U + E_{yrast}(I) - E_{yrast}(I - 1)).$$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<tbody>
<tr>
<td>Level density</td>
<td>a (MeV$^{-1}$)</td>
</tr>
<tr>
<td></td>
<td>GDR $k$</td>
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<tr>
<td></td>
<td>$w_k$</td>
</tr>
<tr>
<td></td>
<td>$E_{Gk}$ (MeV)</td>
</tr>
<tr>
<td></td>
<td>$\Gamma_{Gk}$ (MeV)</td>
</tr>
</tbody>
</table>

Table 2: Input parameters for the GAMBLE simulation of the Dy$^{154}$ decay [13].

In order to test the spin-dependent fitting function, we performed a GAMBLE simulation of the $^{154}$Dy decay, using the program EnerOrd [13] the EOS were constructed, Fig. 11. Four different well defined $(I, E)$ initial states were used. The spin-dependent fitting function ($n = 4$) eq. (25) was numerically computed using the $E_{yrast}(I)$ provided by GAMBLE.

The simulation was performed using the Fermi Gas level density eq. (4) and the GDR gamma strength eq. (2). The parameters describing the $^{154}$Dy decay are listed in Table 2.

By eye (Fig. 11) we can check that the fitting function seems to work well for all cases in the entire energy range, besides the fit is much better than previous attempts [5, 14].

In order to quantify how good is the fit and because the fitting function is a tool to analyze the real spectra (to extract physical parameters), we calculated the fitting function eq. (25) with
Figure 11: GAMBLE simulation and spin-dependent fitting function eq. (25) for different initial states and therefore different initial intrinsic excitation energy $U$. The states are: $(I, E) = (47, 24.6)$ with $U = 6.6$, $(I, E) = (47, 27.0)$ with $U = 9.0$, $(I, E) = (47, 29.0)$ with $U = 11.0$, $(I, E) = (47, 31.0)$ with $U = 13.0$, energy in MeV and spin in $\hbar$. A good fit is obtained in the entire energy range for all cases.

Different level density parameter $a$ near to the simulated one $18.4 \text{ MeV}^{-1}$, and evaluated the $\chi^2_r$ of the fit to the simulated data. Varying $a$ by $0.1 \text{ MeV}^{-1}$, we chose the $a$ value that minimize $\chi^2_r$ like our experimental $a$. As it is shown in Fig. 12, the $\chi^2_r$ of the fit is minimized near the simulated $a$.

The extracted $a$ (Table 3) is very close to the simulated value for all the cases, the small error shows that the present method to extract parameters is very good and that the developed spin-dependent fitting function is good enough.

There are methods to estimate the uncertainty of $a$ from the previous fit process [15], however computational problems in calculating the fitting function and $\chi^2_r$ prevented us from obtaining reliable values.
4.4 Entry distribution $\rho_0(E, I)$

Experimentally we cannot construct a well defined initial state $(I, E)$, because after a fusion-evaporation reaction the nuclei have a wide distribution over the phase space. Even using the Hk technique we can only isolate cascades from a small but finite-size phase-space region. For this reason it is necessary to construct the fitting function for a given initial entry distribution $\rho_0(I, E)$.

Given a distribution $\rho_0(I, E)$ over the initial states the fitting function will be the integration over energy $E$ and the sum over the spin $I$ of the occupied states distribution multiplied by the spin-dependent fitting function eq. (25), that is:

$$p_{N_E=1}^S(E_\gamma; \rho_0(I, E)) = \sum_{I=0}^{N_E} \int_{E_{yrast}(I)} dE \rho_0(I, E)p_{N_E=1}^S(E_\gamma; (I, E)).$$

Replacing eq. (25):

$$p_{N_E=1}^S(E_\gamma; \rho_0(I, E)) = \sum_{I=0}^{N_E} \int_{E_{yrast}(I)} dE \rho_0(I, E)p_{N_E=1}^S(E_\gamma; E - E_{yrast}(I - 1)).$$  \hspace{1cm} (27)
Because \( p_{N_E=1} \) only depends on the effective excitation energy it would be useful to obtain an expression of the form

\[
p_{N_E=1}^{S}(E_\gamma;\rho_0(I,E)) = \int dU \rho_0(U^{\text{eff}}) p_{N_E=1}(E_\gamma, U^{\text{eff}}),
\]

where only an integral over \( U^{\text{eff}} \) has to be performed, and where the distribution over the effective intrinsic excitation energy \( \rho_0(U^{\text{eff}}) \) has been introduced. In order to achieve that, we perform the change of variable in eq. (27) \( U^{\text{eff}} = E - E_\text{yrast}(I-1) \), obtaining:

\[
p_{N_E=1}^{S}(E_\gamma;\rho_0(I,E)) = \sum_{I=0} \int_{E_\text{yrast}(I)-E_\text{yrast}(I-1)} dU^{\text{eff}} \rho_0(I, U^{\text{eff}}+E_\text{yrast}(I-1)) p_{N_E=1}(E_\gamma, U^{\text{eff}}).
\]

If the distribution \( \rho_0(E,I) \) is not so wide in the spin, the term \( E_\text{yrast}(I) - E_\text{yrast}(I-1) \) can be averaged in the spin region of interest, in this way we define:

\[
U_{\text{min}} = (E_\text{yrast}(I) - E_\text{yrast}(I-1)).
\]

Making this approximation the fitting function can be written in the desired form:

\[
p_{N_E=1}^{S}(E_\gamma;\rho_0(I,E)) = \int_{U_{\text{min}}} dU^{\text{eff}} \left( \sum_{I=0} \rho_0(I, U^{\text{eff}}+E_\text{yrast}(I-1)) \right) p_{N_E=1}(E_\gamma, U^{\text{eff}}).
\]

Then the fitting function is:

\[
p_{N_E=1}^{S}(E_\gamma;\rho_0(I,E)) = \int_{U_{\text{min}}} dU^{\text{eff}} \rho_0(U^{\text{eff}}) p_{N_E=1}(E_\gamma, U^{\text{eff}}),
\]

where

\[
\rho_0(U^{\text{eff}}) = \sum_{I=0} \rho_0(I, U^{\text{eff}}+E_\text{yrast}(I-1)).
\]

If we know the entry distribution in terms of \( U \) and \( I; \rho_0(I,U) \), the same procedure can be performed obtaining the same expression eq. (29) with \( \rho_0(U^{\text{eff}}) \) given by:

\[
\rho_0(U^{\text{eff}}) = \sum_{I=0} \rho_0(I, U^{\text{eff}} - E_\text{yrast}(I) + E_\text{yrast}(I-1)).
\]

5 Applications

Previously we have developed and tested the spin-dependent fitting function, obtaining very good results. Now, because the fitting function is an analytical tool to analyze the experimental spectra, we propose some possible applications to hypothetical experimental situations based on previous works. The potential of the developed tool will be shown.
5.1 Hk box

As it was stated previously, setting a gate on \( k \) and \( H \) (that is isolating the cascades which had fired \( k \) detector with a sum of energies between \( H \) and \( H + \Delta H \)) the Hk technique allows us to define a probability density function \( \rho_0(I, E) \) over the initial states for those cascades. In the real experiment it is not possible to construct the EOS for a well defined initial state \((I, E)\), the closest physical region can be built taking cascades from a pair \((k, H)\), what defines the smallest phase space region whose EOS can be experimentally constructed.

For this reason we have to calculate the fitting function for a initial entry distribution \( \rho_0(I, E) \) given by the Hk technique. The detailed functional form of the distribution depends strongly on the experimental setup, in this work we reproduce only the main features of the experimental distributions; the mean value and standard deviation of the projection on \( E \) and \( I \).

![Figure 13](image)

Figure 13: a) Projection on \( E \), b) projection on \( I \) and c) contour plot of the entry Hk-distribution eq. (32) using the parameters of Table 4. d) \( \rho_0(U^{eff}) \) distribution calculated with eq. (31) for the previous entry distribution.

GAMBLE allows to fix the entry population using a distribution originally thought for the residues after fusion-evaporation reactions. The non normalized distribution as function of \( I \) and \( U \) is [13]:

\[
\rho_0(I, U) = \rho(U, I)e^{-\beta_p U^{\alpha_p}} \exp \left( -\frac{(I - \langle I \rangle)^2}{2\sigma_I^2} \right) \tag{32}
\]
where the \( U \) dependence is given by the nuclear level density eq. (4) and a thermal factor 
\[ \exp(-\beta_p U^{\alpha_p}) \]. Besides the term \( 2I + 1 \) in \( \rho(U, I) \), there is gaussian distribution over the spin centered at \( \langle I \rangle \) and with a standard deviation \( \sigma_I \).

\[
\begin{align*}
\beta_p &= 9.32 \times 10^{-10} \\
\alpha_p &= 9 \\
\langle I \rangle &= 44 \\
\sigma_I &= 3.8
\end{align*}
\]

Table 4: GAMBLE parameters for the entry Hk distribution (energy in MeV and spin in \( \hbar \)).

Using the parameters of Table 4 the distribution for a pair \((k, H) = (35, 21 \text{ MeV})\) is reproduced (Fig. 13). The \( E \)-projection's mean value is 25 MeV and its standard deviation is 1.5 MeV, for the \( I \)-projection we obtain \( 44 \hbar \) and \( 3.8 \hbar \) respectively.

Using eq. (32) we calculated the \( U^{\text{eff}} \) distribution with eq. (31). But because the fitting function has to be numerically calculated, we are not interested in obtaining the functional form of \( \rho_0(U^{\text{eff}}) \) but in evaluating it at certain \( U^{\text{eff}} \) values. \( \rho_0(U^{\text{eff}}) \) was calculated for 1 MeV multiples (Fig. 13d). This distribution has important contributions in the range \((7 \text{ MeV}, 13 \text{ MeV})\). Thus the fitting function was calculated with:

\[
p_{N_E=1}^{S} (E_\gamma; \rho_0(I, U)) = \sum_{U^{\text{eff}}=7}^{13} \rho_0(U^{\text{eff}}) p_{N_E=1} (E_\gamma, U^{\text{eff}}), \tag{33}
\]

with energy in MeV. Notice that in this case, the approximation made in eq. (28) is irrelevant because \( U_{\text{min}} \approx 1 \text{ MeV} \) and there are no contributions at such energy.

![Figure 14: GAMBLE simulation and fitting function calculated using a) the mean value of \( U^{\text{eff}} \) and using b) \( \rho_0(U^{\text{eff}}) \).](image)

The fitting function was calculated with eq. (33) obtaining an excellent fit to the simulation (Fig. 14b), we also computed the fitting function using only the average value of \( U^{\text{eff}} \) obtaining a
no very good fit, hence it is not enough to use only the average value. The level density parameter \( a \) was extracted by minimizing \( \chi^2 \) from the fit of eq. (33) to the simulated data, we obtained \( a = 18.7 \text{ MeV}^{-1} \). It suggests that the developed fitting function blended with the Hk technique can be used to analyze experimental EOS considering a non-ideal response function.

### 5.2 Nuclear phase transition

Theoretical models [16] predict the existence of different phase space regions characterized by different nuclear deformation, thus for \(^{154}\text{Dy}\) there are prolate and oblate deformation regions, each one with different level density parameter \( a \). Now we want to investigate if the Hk-EOS method and the developed fitting function would allow to study the phase transitions, that means to distinguish the two regions with different \( a \) and to determine their values.

![Figure 15: a) Level density parameter \( a \) as a function of the intrinsic excitation energy, eq. (34). b) Scheme of the two deformation regions on the phase-space generated by eq. (34).](image)

We are going to consider the case of the level density parameter depending on \( U \), we assume a Fermi function for \( a(U) \) (Fig. 15b):

\[
a(U) = \left( 18.4 + 3.7 \frac{1}{e^{8(U-5)} + 1} \right) \text{ MeV}^{-1},
\]

with \( U \) in MeV. It defines two different regions in the phase-space with level density parameters \( a_1 = 18.4 \text{ MeV}^{-1} \) and \( a_2 = 22.1 \text{ MeV}^{-1} \) as is shown in Fig.15b.

GAMBLE does not enable to fix different \( a \) regions as given by eq. (34). We showed before that the spin-independent case and the spin-dependent case are not very different, therefore we will use the spin-independent simulation to study the phase transitions described above.

The spin-independent simulation for the \(^{154}\text{Dy}\) decay was performed using the Fermi Gas level density eq. (4) and the GDR eq. (2) with the parameters of Table 2 and \( a \) given by eq. (34). We fixed the initial intrinsic excitation energy at 11 MeV.

Fig. 16a shows the simulated EOS and the spin-independent fitting function \( p_{N_k=1} \) with \( a_1 \) and \( a_2 \). There are two peaks in the simulated EOS, one at 4 MeV and other at 6 MeV. The first one is similar to the peak of the spectra with constant \( a \) while the second one gives a spectral
shape never observed in previous simulations, therefore it is due to the phase transition and for this reasons we call it the transition peak.

\( \gamma \)-energies above the transition peak (\( \approx 6 \) MeV) are transitions whose final states have \( U_f < (11 - 6) \) MeV= 5 MeV which is the energy at the phase transition, therefore those \( \gamma \)-rays depend on \( a_2 \). On the other hand \( \gamma \)-energies below the peak may be produced by transitions to final states in both regions. Because of the transition peak, we can conclude that the EOS allows to distinguish the two regions, and the intrinsic excitation energy at the transition

In Fig. 16a, above the second peak the simulated spectrum and the fitting function with \( a_2 \) are the same but with a shift, recalling that the plot is in semi-logarithmic scale the shift can be corrected by a normalization factor. Below the transition peak in a small energy range, the spectrum is the fitting function with \( a_1 \) multiplied by a normalization constant. In Fig. 16b the fitting function for \( a_1 \) and \( a_2 \) were multiplied by the appropriate normalization factors, it shows that the simulated spectrum is fitted well above the transition peak by \( p_{N_E=1} \) for \( a_2 \) and in a small range below the peak for \( a_1 \). It proves that it is possible to extract the level density parameter for both regions using the developed fitting function.

In Fig. 16b, there is an energy interval (5.5 MeV, 6.5 MeV) between the two regions where the simulated spectrum is fitted well by \( p_{N_E=1} \) for \( a_1 \) and \( a_2 \); it is the interval (\( \approx 1 \) MeV) at which the change between \( a_1 \) and \( a_2 \) occurs in \( a(U) \). Hence the transition interval can be extracted from the fit.

Finally, knowing the behavior of \( a(U) \), the fitting function can be computed using \( a \) from eq. (34) in \( f(E_i,U_i) \). In Fig. 17 the fitting function reproduces well the simulated spectrum in the entire energy range.
6 Conclusions

The spin-independent fitting function was developed eqs. (21, 22) and tested with very good results, Fig. 8, 9. Furthermore making some assumptions, the fitting function allowed to fit well the GAMBLE spectra (spin-dependent case). By minimizing $\chi^2$ we can extract the level density parameter $a$ from the fit to the simulated spectra (Table 3) obtaining a good agreement with the input value of the simulation.

The good results of Fig. 14 for an entry population given by the Hk technique show that, considering a non-ideal response function of the detector array, the developed fitting function can be used to analyze the experimental spectra obtained with Hk-EOS.

In the case of $a(U)$ generating two phase-space regions Fig. 15, the fitting function enabled to determine the $a$ for both regions and the intrinsic excitation energy at the transition. Also the entire spectrum can be fitted.

7 Perspectives

Because the fitting function has to be numerically computed, the programs and codes have to be improved in order to obtain a fitting function more accurate and applicable to data analysis, allowing the extraction of parameters and its uncertainties form the fit to the spectra.

GAMBLE has to be modified in order to study the case of $a(U)$ and $a(I)$, looking for new applications of the developed fitting function.

Figure 17: Simulated EOS and fitting function $p_{N_E=1}$ for $a(U)$ in eq. (34). A good fit in the entire energy range can be observed.
8 Acknowledgments

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